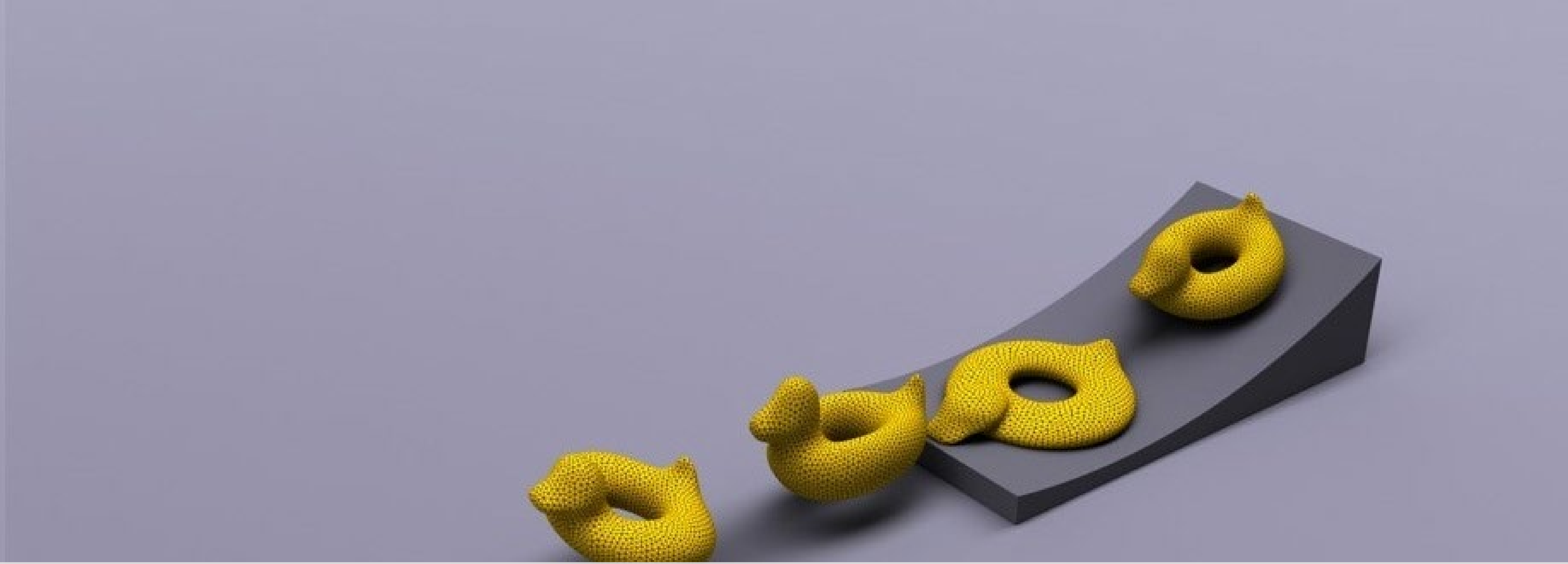


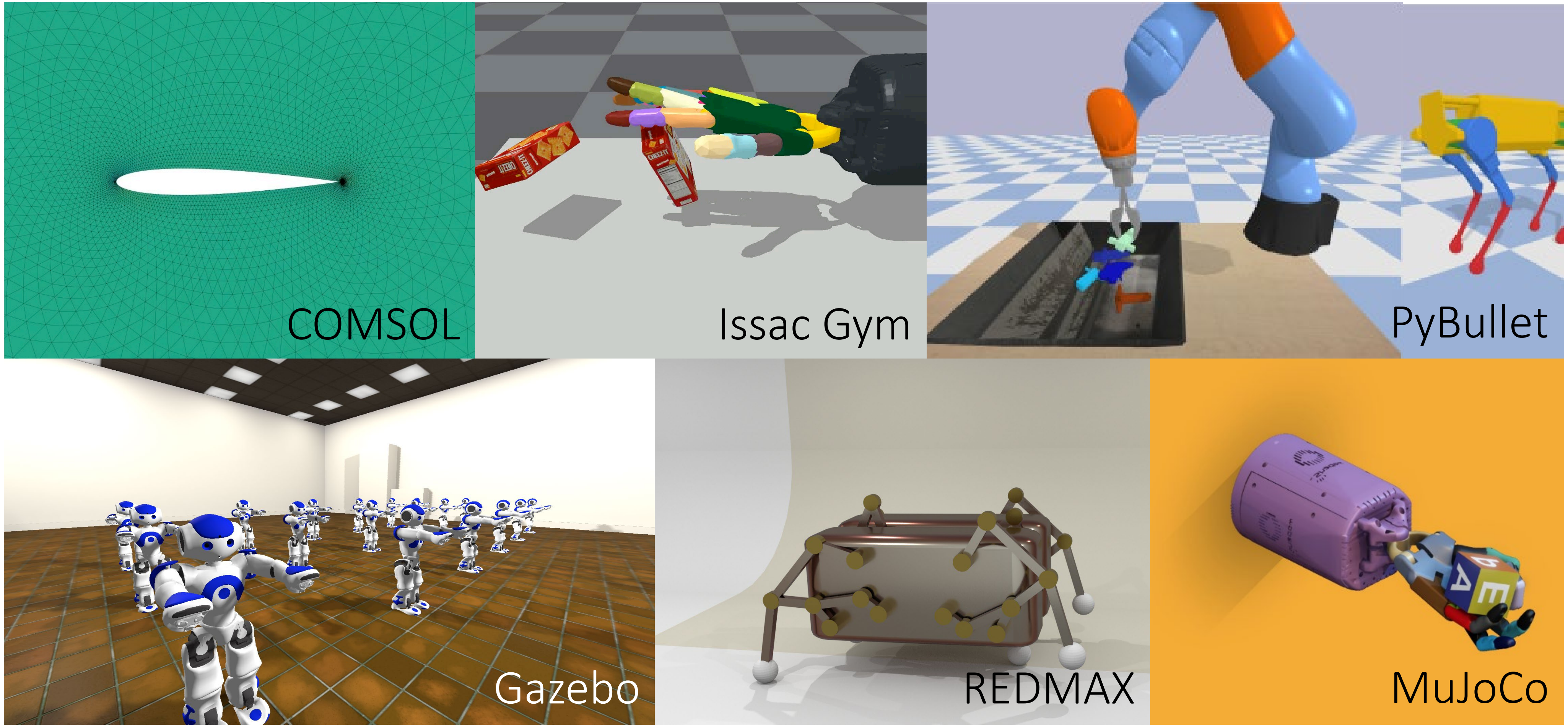
DiffPD: Differentiable Projective Dynamics

Tao Du, Kui Wu, Pingchuan Ma, Sebastien Wah, Andrew Spielberg,

Daniela Rus, Wojciech Matusik

MIT CSAIL

Simulation platforms for learning and robotics

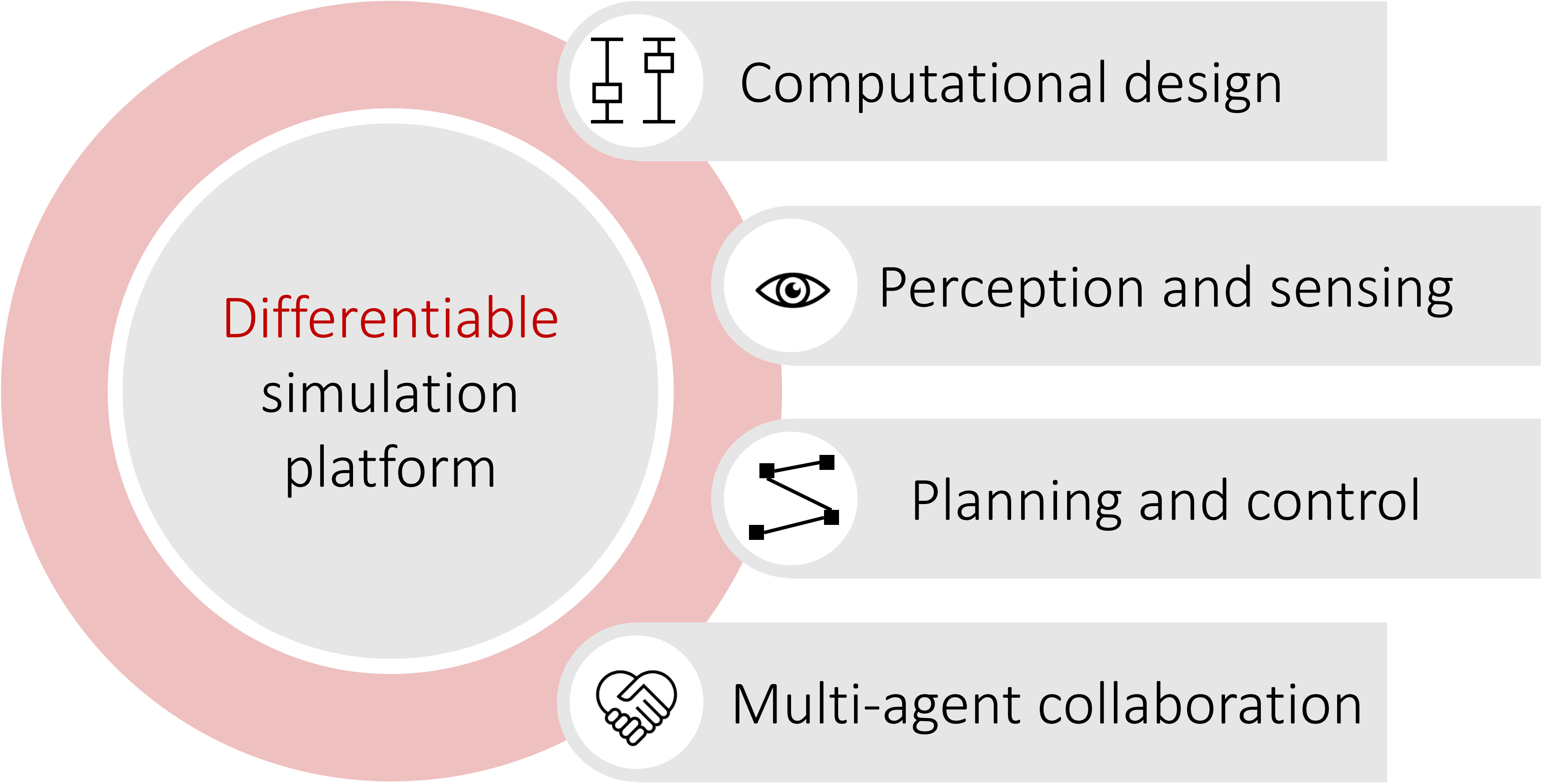


Simulation platforms for learning and robotics

Simulation platform

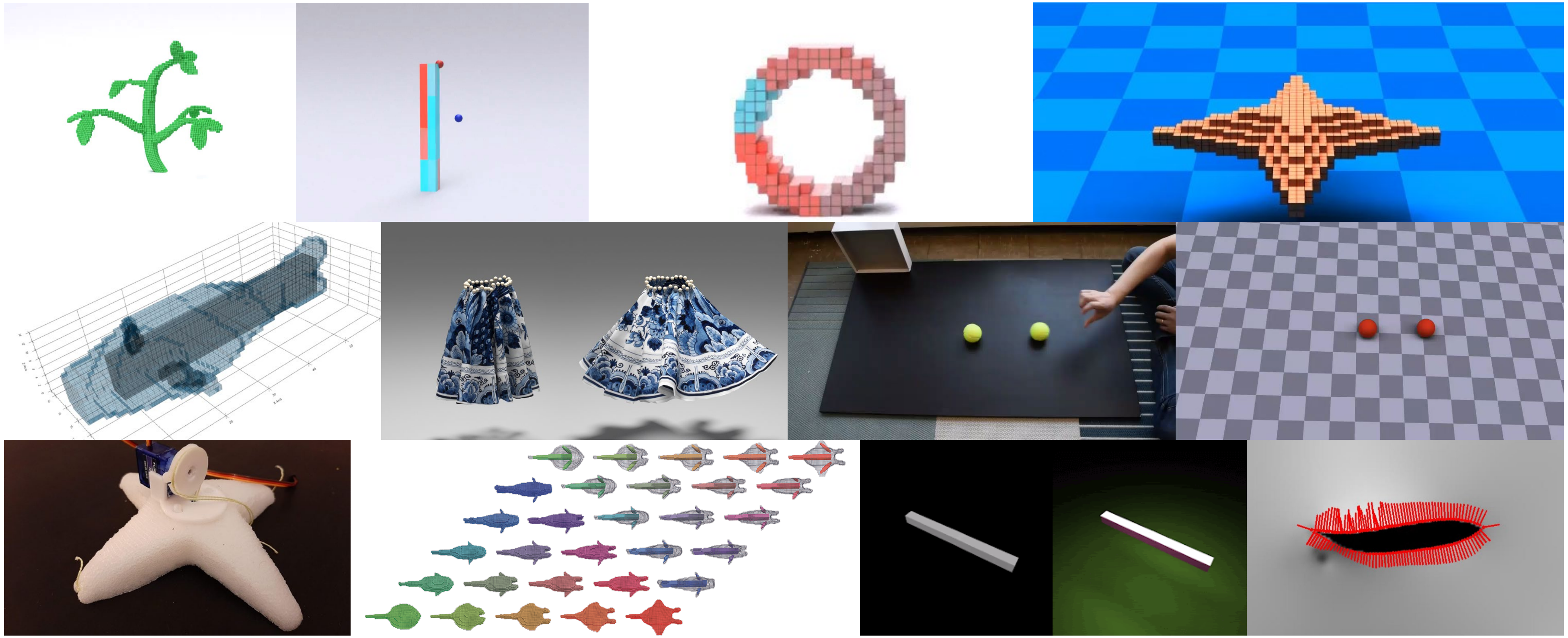
- Computational design
- Perception and sensing
- Planning and control
- Multi-agent collaboration

Simulation platforms for learning and robotics



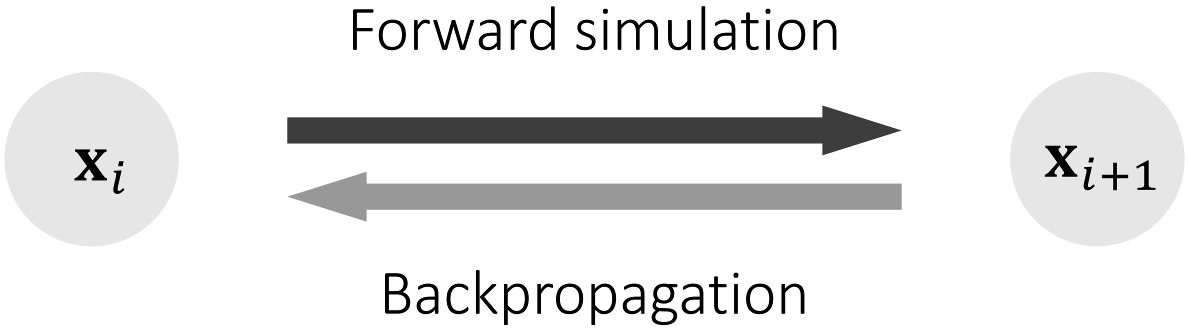
DiffPD: a differentiable soft-body simulator

A simulator that unlocks interesting downstream applications.



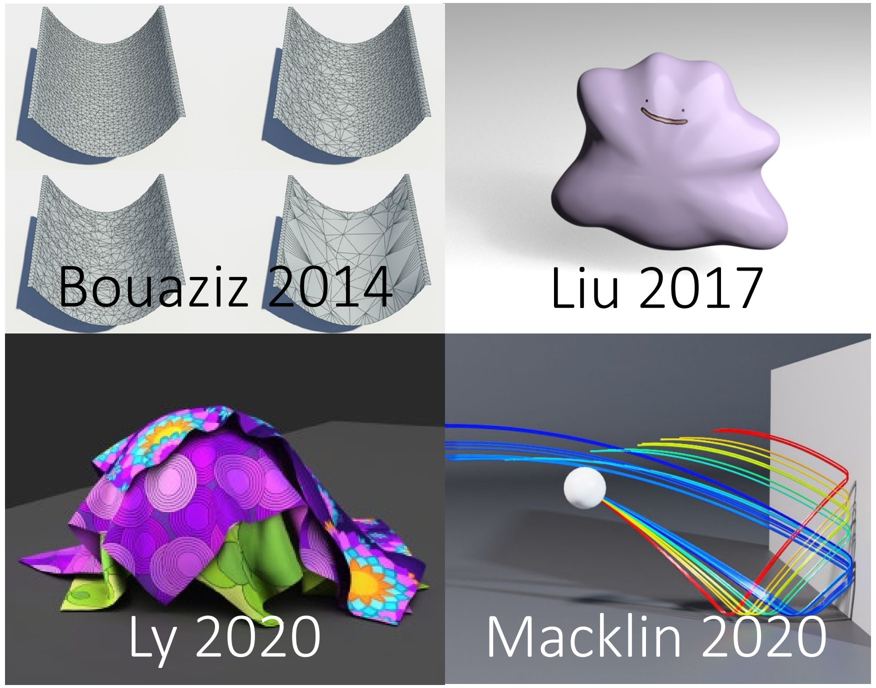
DiffPD: a differentiable soft-body simulator

A simulator that reveals interesting mathematical insights for developing differentiable simulators (more on this later).



Related work

Soft-body simulation



Differentiable physics

Method

Background: implicit time integration

Consider the i-th step of simulation with timestep h.

Input: nodal position $\mathbf{x}_i$ and velocity $\mathbf{v}_i$.

Output: new nodal position $\mathbf{x}_{i+1}$.

$$\mathbf{x}_{i+1} = \mathbf{x}_i + h\mathbf{v}_{i+1}$$
$$\mathbf{v}_{i+1} = \mathbf{v}_i + h\mathbf{M}^{-1}[-\nabla E(\mathbf{x}_{i+1}) + \mathbf{f}_{\text{ext}}]$$

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mass matrix

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internal force from elastic energy

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mass matrix

internal force from elastic energy

external force

Background: implicit time integration

Recast it as a saddle-point problem: find $\nabla g(\mathbf{x}_{i+1}) = \mathbf{0}$ where

$$g(\mathbf{x}) \coloneqq \frac{1}{2h^2}(\mathbf{x}-\mathbf{y})^\top \mathbf{M}(\mathbf{x}-\mathbf{y}) + E(\mathbf{x})$$

Background: implicit time integration

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$\mathbf{y} \coloneqq \mathbf{x}_i + h\mathbf{v}_i + h^2\mathbf{M}^{-1}\mathbf{f}_{\mathrm{ext}}$ is independent of $\mathbf{x}$.

Background: implicit time integration

Newton's method: $\mathbf{x}^{k+1} = \mathbf{x}^k + \Delta\mathbf{x}^k$ where

$$\nabla^2 g(\mathbf{x}^k)\Delta\mathbf{x}^k = \nabla g(\mathbf{x}^k)$$

Bottleneck: solving the matrix $\nabla^2 g(\mathbf{x}^k)$:

$$\nabla^2 g(\mathbf{x}^k) = \frac{1}{h^2}\mathbf{M} + \nabla^2 E(\mathbf{x}^k)$$

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$$\nabla^2 g(\mathbf{x}^k) = \frac{1}{h^2}\mathbf{M} + \nabla^2 E(\mathbf{x}^k)$$

requires recomputation whenever $\mathbf{x}^k$ changes!

Background: differentiable simulation

Consider backpropagating loss L in the i-th step of simulation.

$$\frac{\partial L}{\partial \mathbf{y}} = \frac{\partial L}{\partial \mathbf{x}_{i+1}} \frac{\partial \mathbf{x}_{i+1}}{\partial \mathbf{y}}$$

Recall that $\nabla g(\mathbf{x}_{i+1}) = \mathbf{0}$. By differentiating it w.r.t. $\mathbf{y}$ we have

$$\frac{\partial \mathbf{x}_{i+1}}{\partial \mathbf{y}} = \frac{1}{h^2} [\nabla^2 g(\mathbf{x}_{i+1})]^{-1} \mathbf{M}$$

Background: differentiable simulation

Putting everything together, we have $\frac{\partial L}{\partial \mathbf{y}} = \frac{1}{h^2}\mathbf{z}^\top \mathbf{M}$ where

$$\nabla^2 g(\mathbf{x}_{i+1})\mathbf{z} = \left(\frac{\partial L}{\partial \mathbf{x}_{i+1}}\right)^\top$$

Background: differentiable simulation

Putting everything together, we have $\frac{\partial L}{\partial \mathbf{y}} = \frac{1}{h^2}\mathbf{z}^\top \mathbf{M}$ where

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We see that solving $\nabla^2 g$, again, is the bottleneck.

Recap

Forward simulation: $\nabla^2 g(\mathbf{x}^k)\Delta\mathbf{x}^k = \nabla g(\mathbf{x}^k)$.

Backpropagation: $\nabla^2 g(\mathbf{x}_{i+1})\mathbf{z} = \left(\frac{\partial L}{\partial \mathbf{x}_{i+1}}\right)^{\top}$.

Insight

Forward and backward computation share the same bottleneck.

Forward simulation: $\nabla^2 g(\mathbf{x}^k)\Delta\mathbf{x}^k = \nabla g(\mathbf{x}^k)$.

Backpropagation: $\nabla^2 g(\mathbf{x}_{i+1})\mathbf{z} = \left(\frac{\partial L}{\partial \mathbf{x}_{i+1}}\right)^\top$.

Insight

Efficient solvers for forward simulation exist.

Efficient forward simulation: $\nabla^2 g(\mathbf{x}^k)\Delta\mathbf{x}^k = \nabla g(\mathbf{x}^k)$.

Backpropagation: $\nabla^2 g(\mathbf{x}_{i+1})\mathbf{z} = \left(\frac{\partial L}{\partial \mathbf{x}_{i+1}}\right)^\top$.

Insight

Can we borrow them to build efficient backpropagation solver as well?

Efficient forward simulation: $\nabla^2 g(\mathbf{x}^k)\Delta\mathbf{x}^k = \nabla g(\mathbf{x}^k)$.

Efficient backpropagation: $\nabla^2 g(\mathbf{x}_{i+1})\mathbf{z} = \left(\frac{\partial L}{\partial \mathbf{x}_{i+1}}\right)^\top$.

Simulation speedup: Projective Dynamics (PD)

Consider a special class of $E = \sum_c E_c$ where

$$E_c(\mathbf{x}) \coloneqq \min_{\mathbf{p}_c \in \mathcal{M}_c} ||\mathbf{G}_c\mathbf{x} - \mathbf{p}_c||_2^2$$

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energy on each
finite element

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local feature, e.g., deformation gradient

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energy on each finite element

local feature, e.g., deformation gradient

projection of local feature

Simulation speedup: Projective Dynamics (PD)

The saddle-point problem $\nabla g = \mathbf{0}$ is now modified accordingly:

$$\min_{\mathbf{x}, \{\mathbf{p}_c \in \mathcal{M}_c\}} \tilde{g}(\mathbf{x}, \{\mathbf{p}_c\})$$

where

$$\tilde{g}(\mathbf{x}, \{\mathbf{p}_c\}) := \frac{1}{2h^2}(\mathbf{x} - \mathbf{y})^\top \mathbf{M}(\mathbf{x} - \mathbf{y}) + \sum_c ||\mathbf{G}_c \mathbf{x} - \mathbf{p}_c||_2^2$$

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The global step: fix $\mathbf{p}_c$ and solve $\mathbf{x}$, constant matrix in the quadratic form and can be prefactorized.

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The global step: fix $\mathbf{p}_c$ and solve $\mathbf{x}$, constant matrix in the quadratic form and can be prefactorized.

The local step: fix $\mathbf{x}$ and solve $\mathbf{p}_c$, parallelizable among elements.

Backpropagation speedup: DiffPD

With PD, $\nabla^2 g$ becomes

$$\nabla^2 g(\mathbf{x}) = \frac{1}{h^2}\mathbf{M} + \sum_c \mathbf{G}_c^\top \mathbf{G}_c - \sum_c \mathbf{G}_c^\top \frac{\partial \mathbf{p}_c}{\partial \mathbf{x}}$$

$$\coloneqq \mathbf{A} - \Delta\mathbf{A}$$

Backpropagation speedup: DiffPD

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constant matrix and
source of efficiency

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$$\coloneqq \mathbf{A} - \Delta\mathbf{A}$$

constant matrix and source of efficiency

residual that can be computed in parallel.

Backpropagation speedup: DiffPD

Recall the bottleneck:

$$\nabla^2 g(\mathbf{x}_{i+1})\mathbf{z} = \mathbf{b} := \left(\frac{\partial L}{\partial \mathbf{x}_{i+1}}\right)^{\top}$$

With $\nabla^2 g = \mathbf{A} - \Delta\mathbf{A}$ it becomes $(\mathbf{A} - \Delta\mathbf{A})\mathbf{z} = \mathbf{b}$, or

$$\mathbf{A}\mathbf{z} = \mathbf{b} + \Delta\mathbf{A}\mathbf{z}$$

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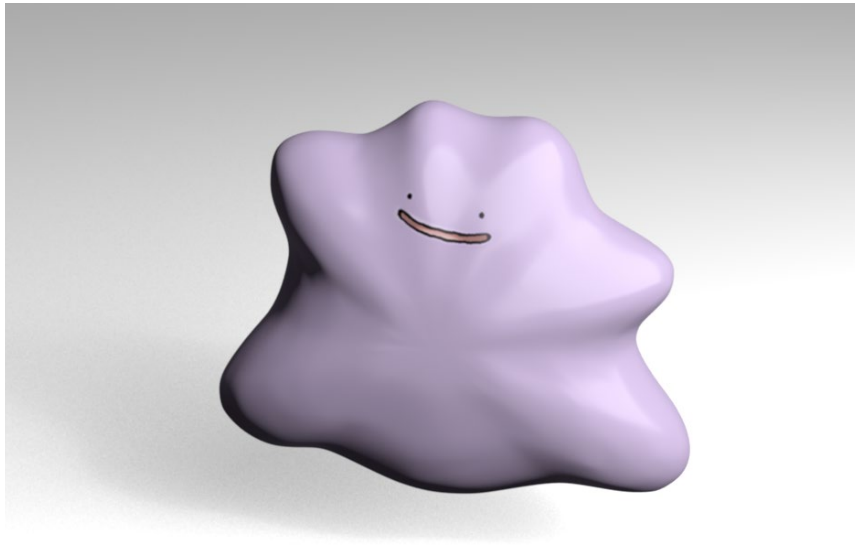
The global step: constant $\mathbf{A}$, already factorized.

The local step: parallelizable on elements.

Extension one: quasi-Newton speedup

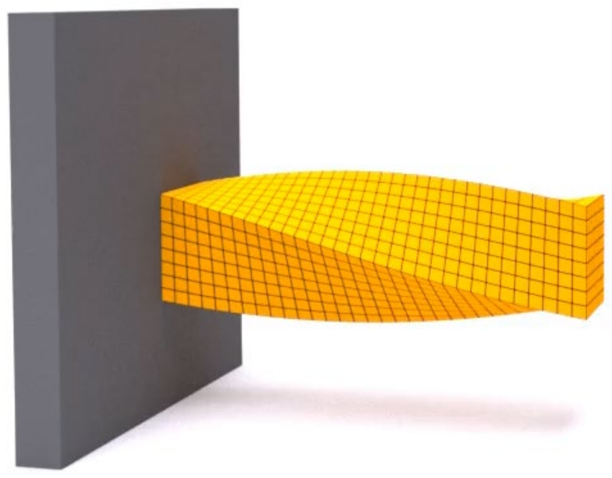
We recall that Liu [2017] proposed a quasi-Newton approach to speed up PD even more. We can transfer it to backpropagation too.

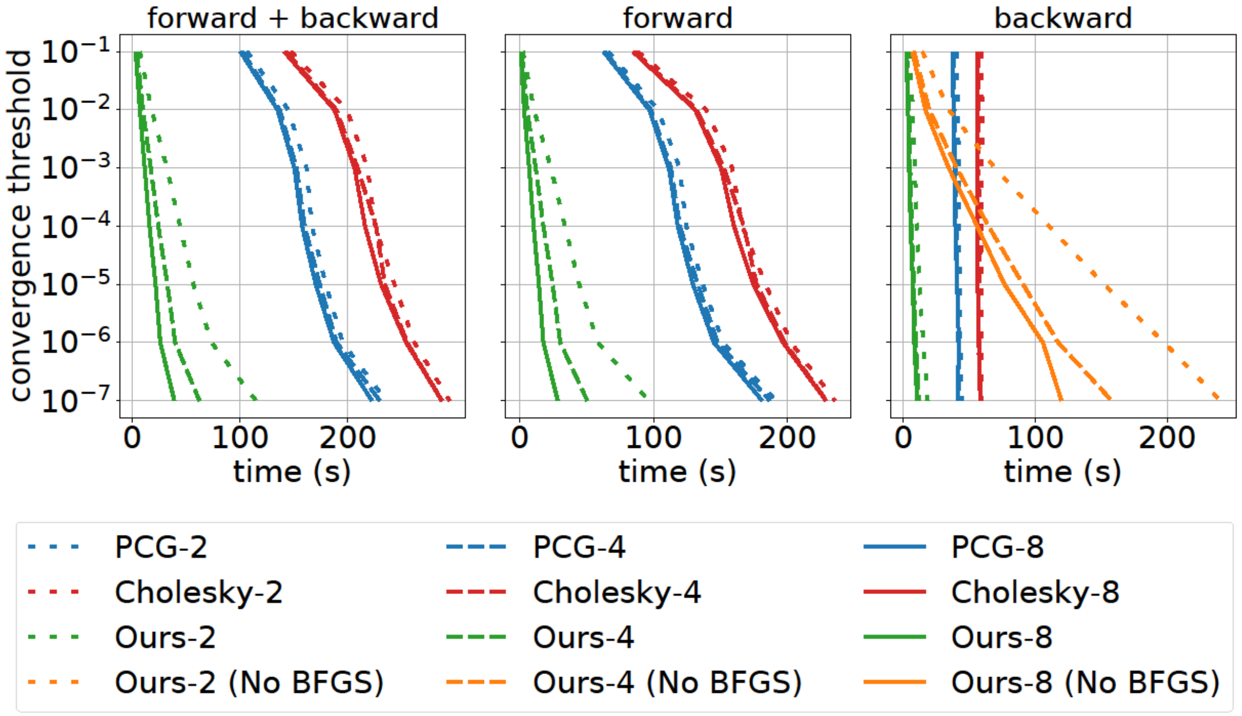


Liu et al. *Quasi-Newton Methods for Real-time Simulation of Hyperelastic Materials*. TOG 2017

Extension one: evaluation

Cantilever (8019 DoFs, 25 steps, 10ms timestep, no contact)





Extension two: boundary conditions

Consider the global step in PD:

$$\mathbf{A}\mathbf{x}_{i+1} = \text{right-hand side from local step.}$$

Efficient solvers exist for $\mathbf{A}$ with erased rows and columns:

$$(\mathbf{A} + \mathbf{u}\mathbf{v}^\top)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{u}(\mathbf{A}^{-1}\mathbf{v})^\top}{1 + \mathbf{v}^\top\mathbf{A}^{-1}\mathbf{u}}$$

Extension two: boundary conditions

Let’s take a look at its source of efficiency:

$$(\mathbf{A} + \mathbf{u}\mathbf{v}^\top)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{u}(\mathbf{A}^{-1}\mathbf{v})^\top}{1 + \mathbf{v}^\top\mathbf{A}^{-1}\mathbf{u}}$$

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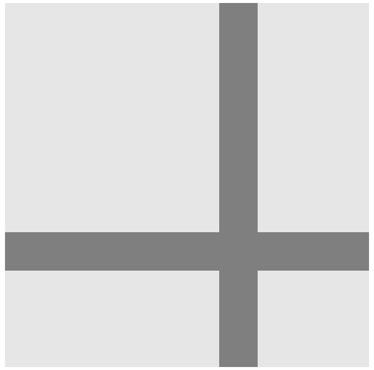

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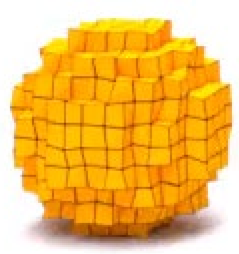
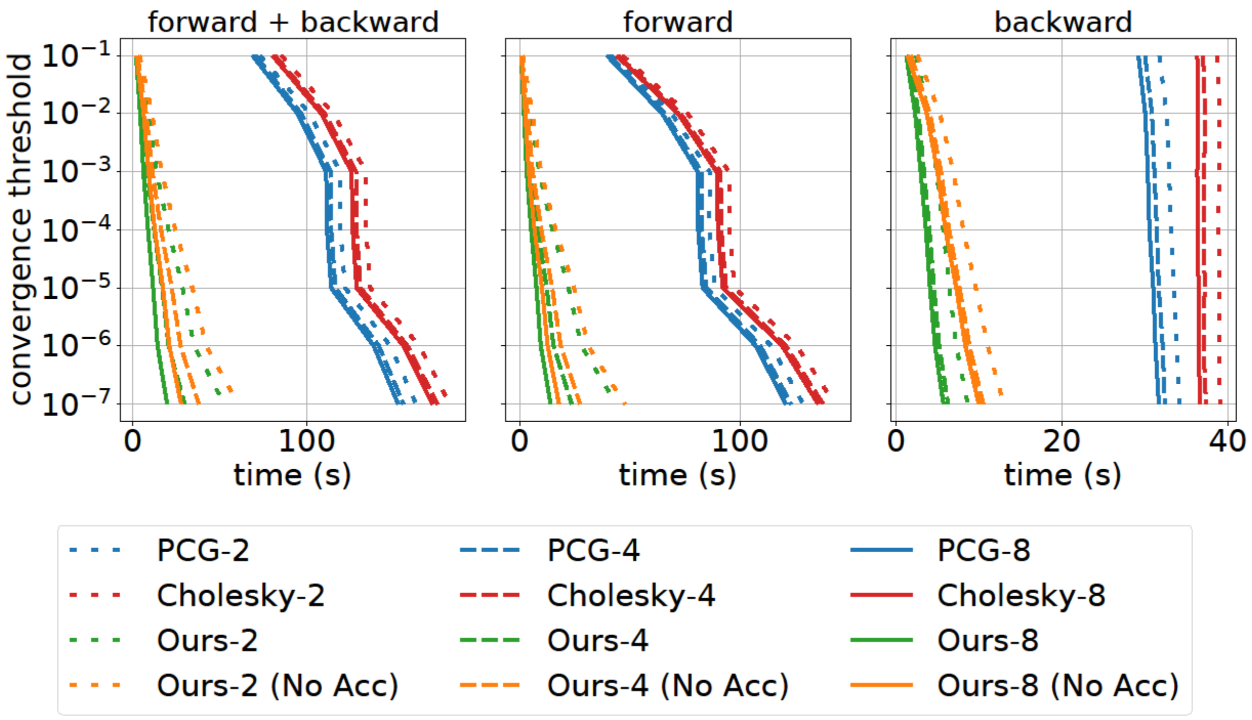
It turns out that we can transfer this idea to backpropagation too, which DiffPD used for contact handling.

Extension two: evaluation

Rolling sphere (2469 DoFs, 100 steps, 5ms timestep, **with contact**)





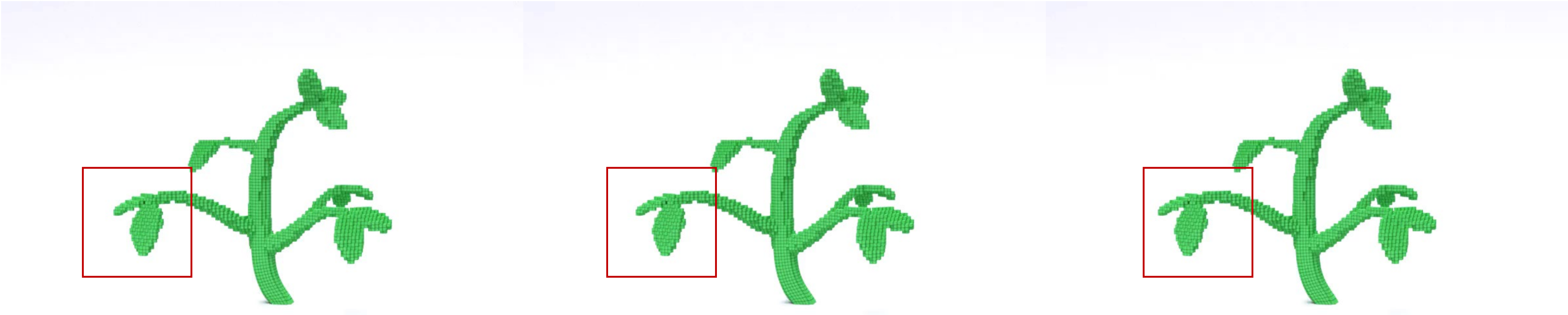
Insight, reiterated

Efficient forward simulation solvers can be transferred to efficient backpropagation solvers!

Applications

System identification

Goal: estimating the material parameters of a plant from its motion.



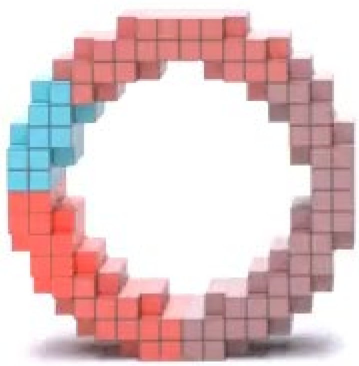
Input (ground truth) Initial guess After optimization

Initial state optimization

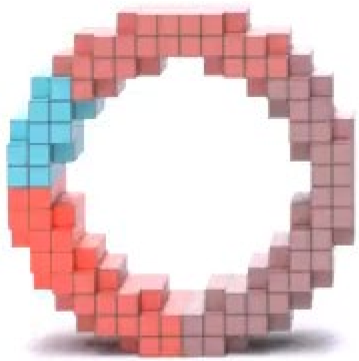
Goal: optimizing time-invariant actuation to reach the target.

Open-loop control

Goal: optimizing actuation to roll forward.



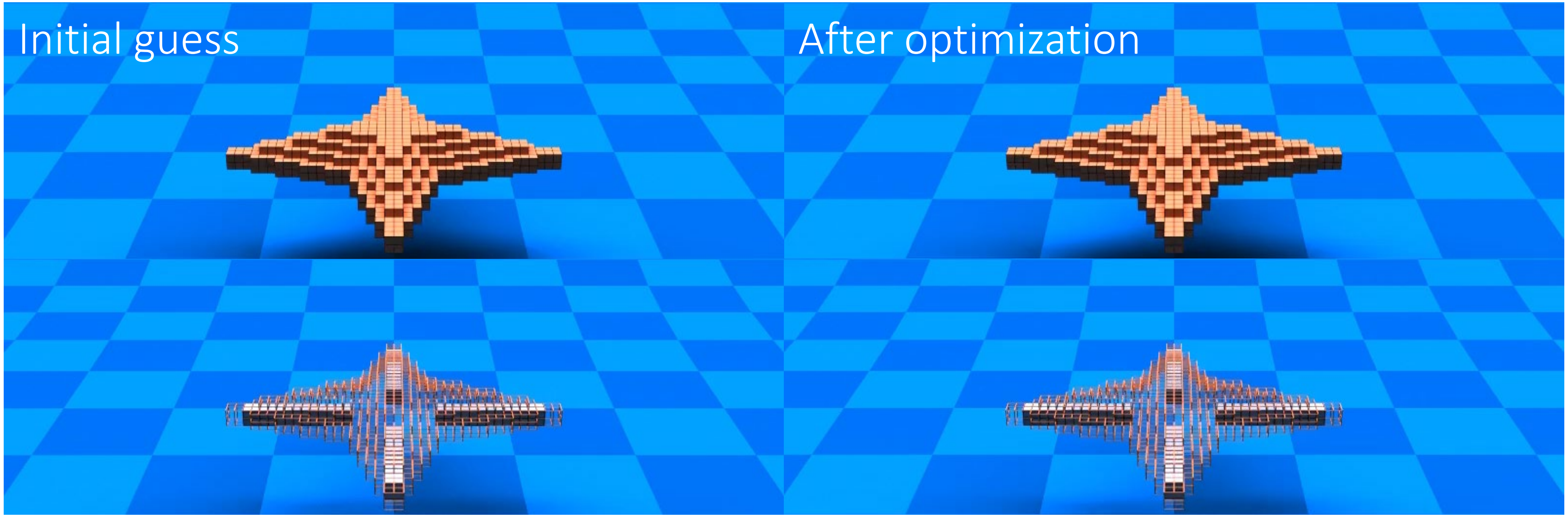
Initial guess



After optimization

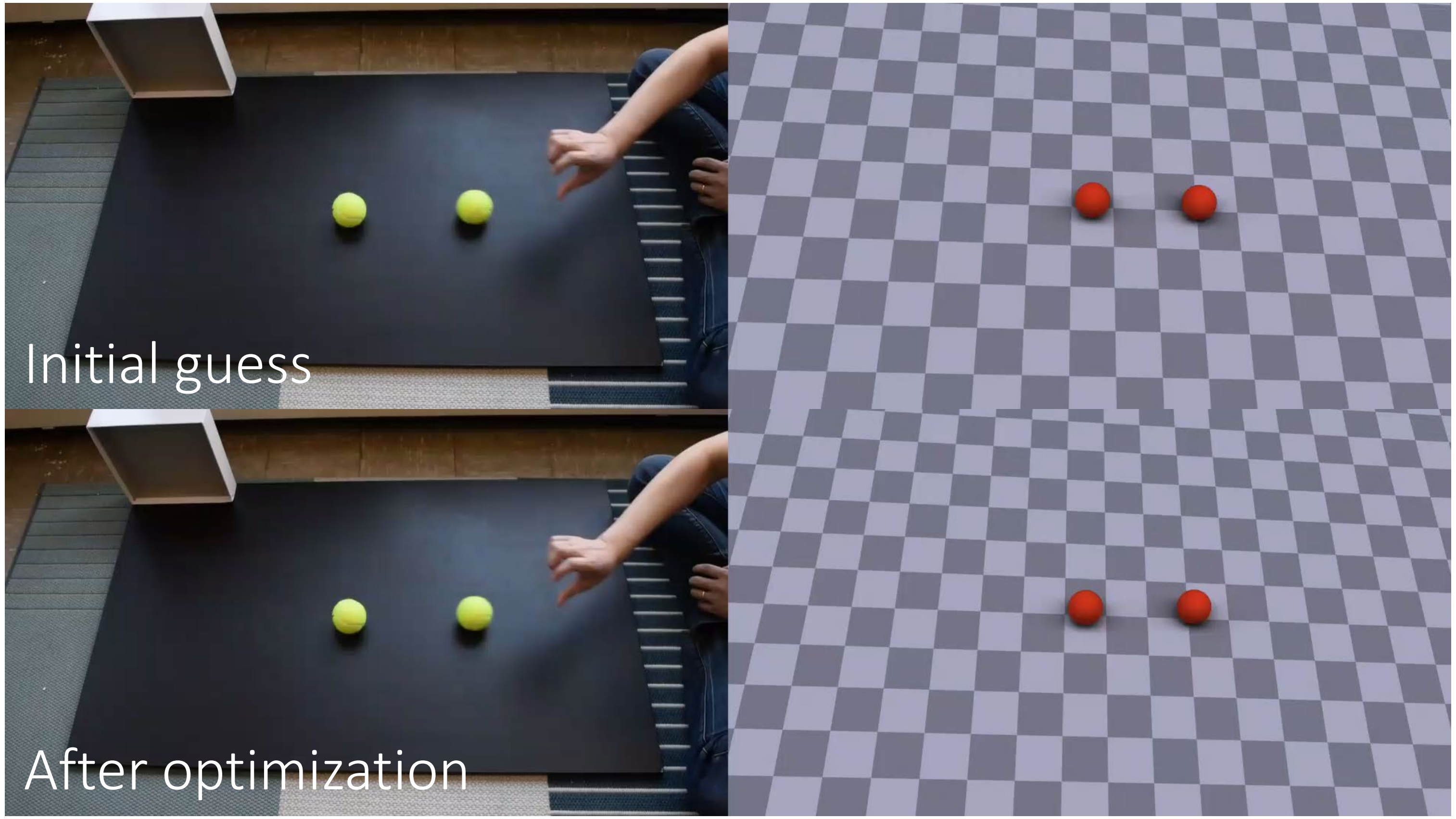
Closed-loop control

Goal: optimizing a neural network controller so that the starfish rises.



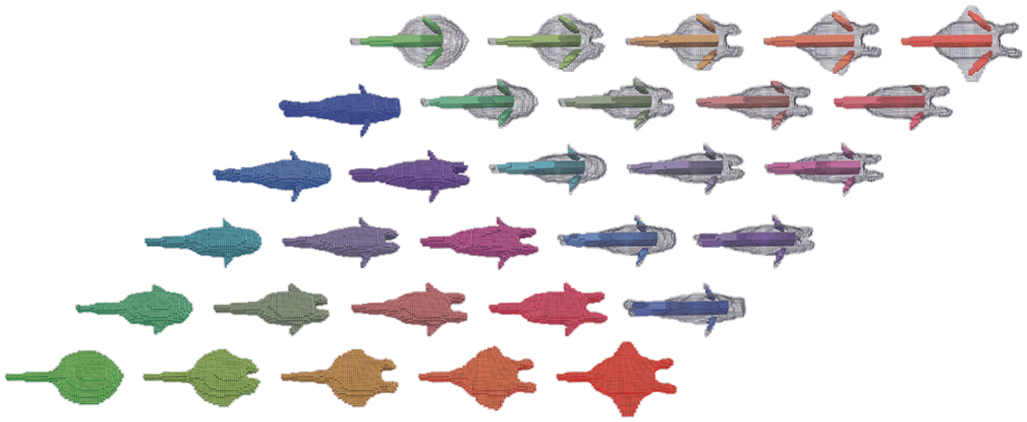
Real-to-sim transfer

Goal: estimating scene parameters to reconstruct the balls' motion.



User gallery: computer graphics

DiffPD are used in computational design of soft characters and cloth.



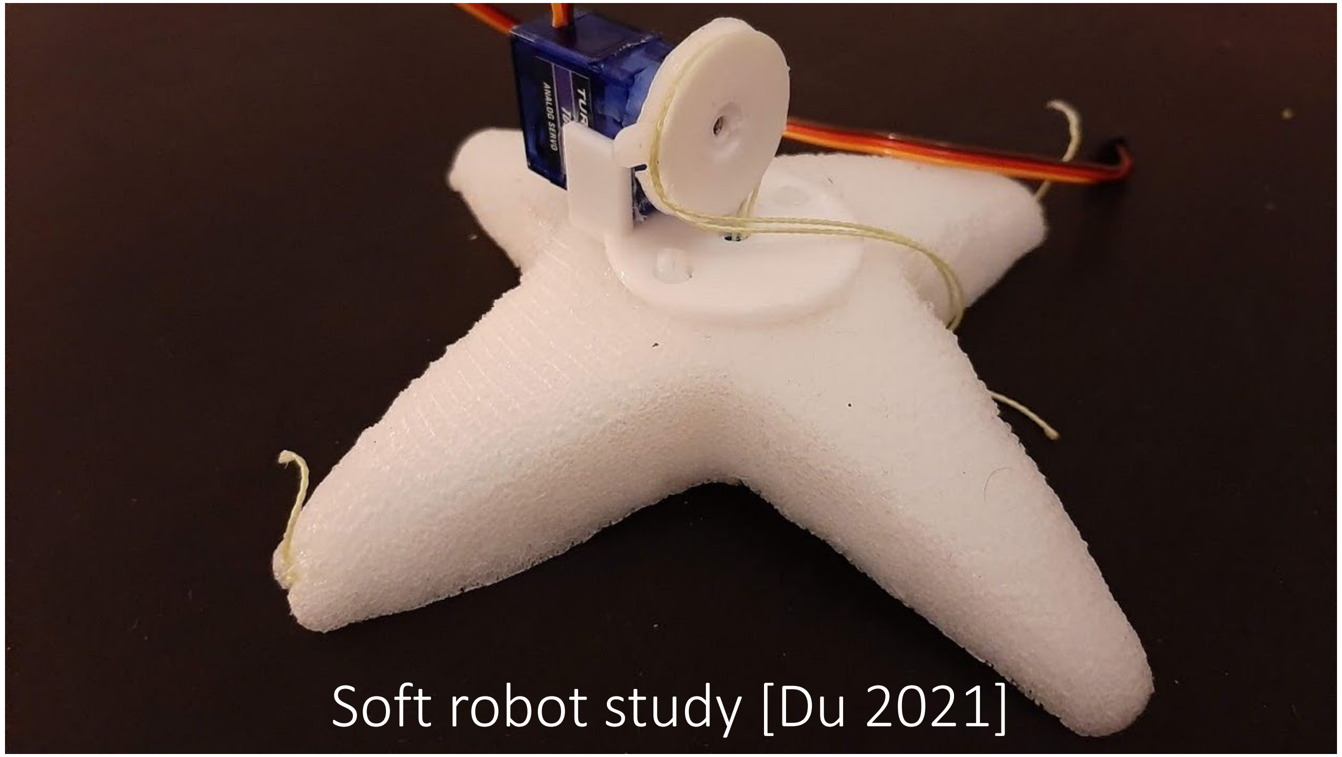
Design of soft underwater characters [Ma 2021]

Cloth simulation [Li 2022]

Ma et al. *DiffAqua: A Differentiable Computational Design Pipeline for Soft Underwater Swimmers with Shape Interpolation*. SIGGRAPH 2021
Li et al. *DiffCloth: Differentiable Cloth Simulation with Dry Frictional Contact*. TOG 2022

User gallery: robotics

DiffPD has also been used in modeling and controlling soft robots.

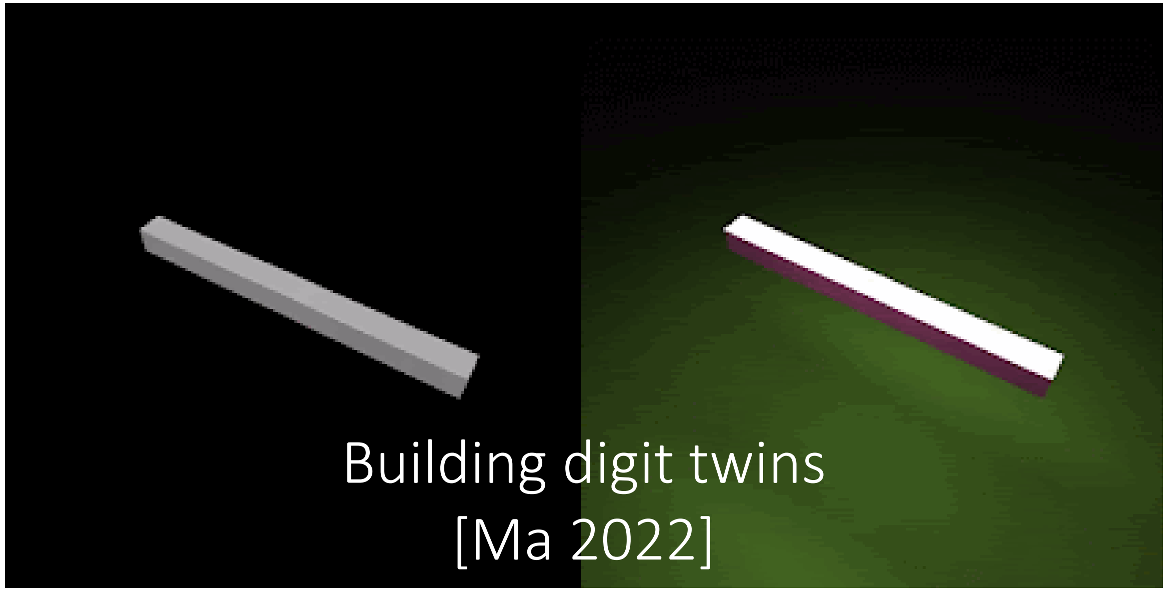


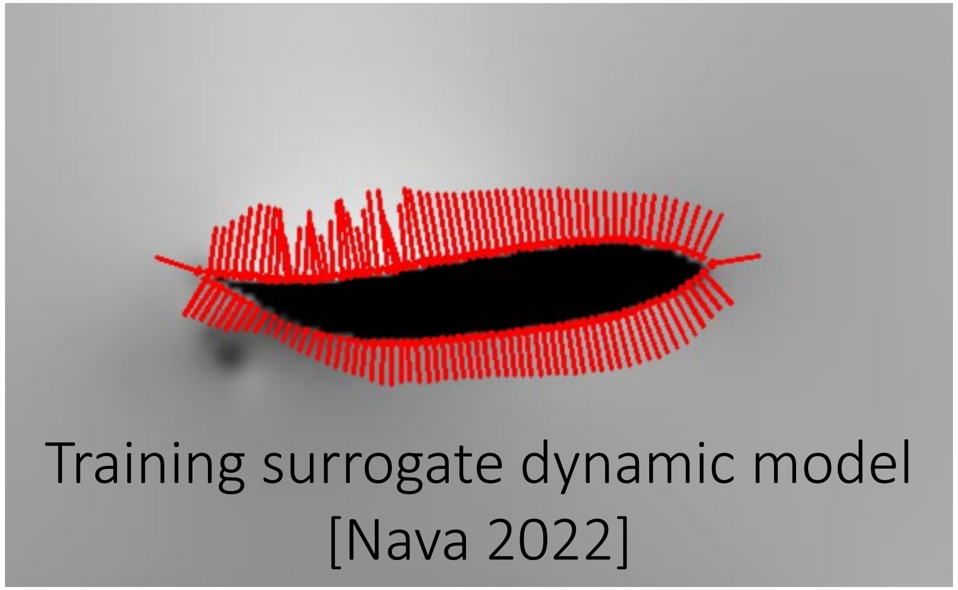
Soft robot study [Du 2021]

Du et al. *Underwater Soft Robot Modeling and Control with Differentiable Simulation*. RA-L 2021

User gallery: machine learning

DiffPD also attracts users from the learning community.





Ma et al. *RISP: Rendering-Invariant State Predictor with Differentiable Simulation and Rendering for Cross-Domain Parameter Estimation*. ICLR 2022
Nava et al. *Fast Aquatic Swimmer Optimization with Differentiable Projective Dynamics and Neural Network Hydrodynamic Models*. ICML 2022

Summary

Conclusions

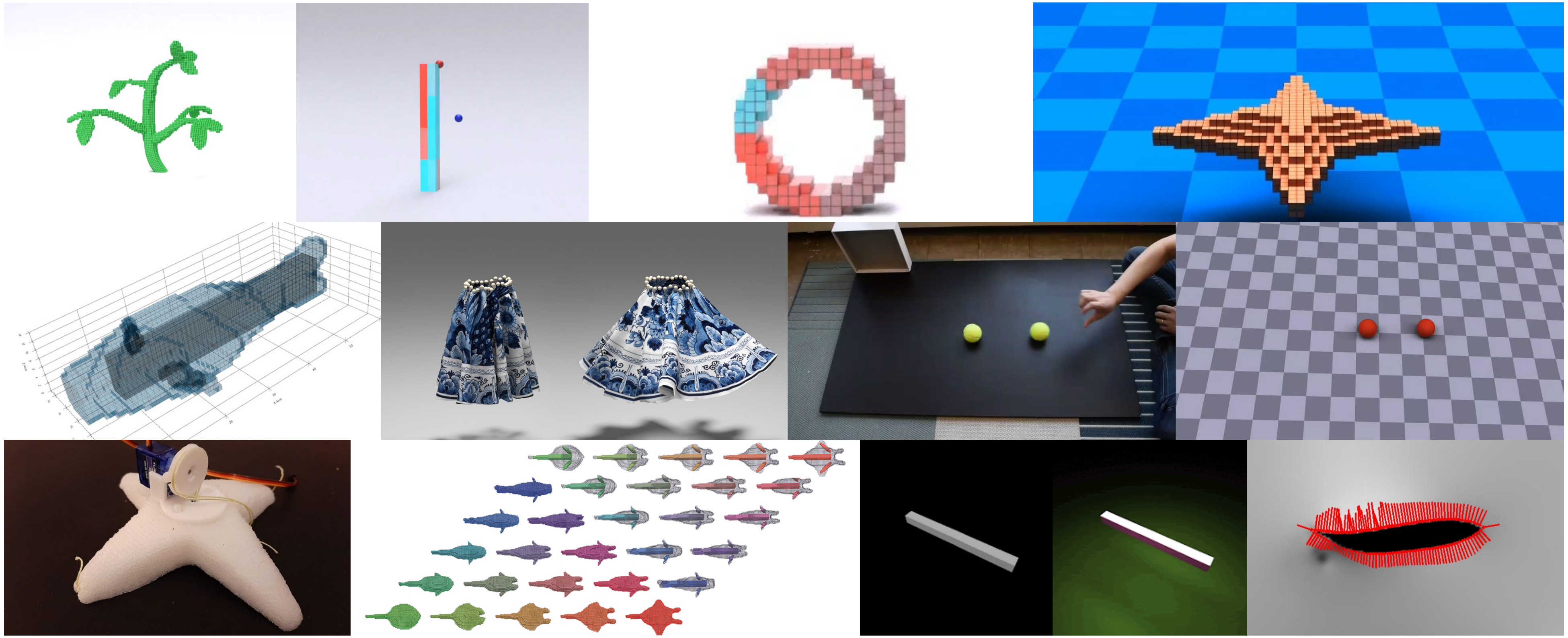
Numerical techniques in forward simulation and backpropagation are two sides of the same coin.

Efficient forward simulation: $\nabla^2 g(\mathbf{x}^k)\Delta\mathbf{x}^k = \nabla g(\mathbf{x}^k)$.

Efficient backpropagation: $\nabla^2 g(\mathbf{x}_{i+1})\mathbf{z} = \left(\frac{\partial L}{\partial \mathbf{x}_{i+1}}\right)^{\top}$.

Conclusions

A fast, reliable differentiable soft-body simulator unlocks wide application in graphics, robotics, and machine learning.



For more information

Project

http://diffpd.csail.mit.edu/

Code

https://github.com/mit-gfx/diff_pd_public

This work is sponsored by DARPA FA8750-20-C-0075, IARPA 2019-19020100001, and NSF 2106962.